

FERMIONIC GLUING PRINCIPLE OF THE TOPOLOGICAL VERTEX

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ABSTRACT. We will establish the fermionic gluing principle of the topological vertex, that is, provided the framed ADKMV conjecture, the generating functions of the Gromov-Witten invariants of all toric Calabi-Yau threefolds are Bogoliubov transforms of the vacuum.

1. INTRODUCTION

In general it is an unsolved problem to compute the Gromov-Witten invariants of an algebraic variety in arbitrary genera. However, in the case of toric Calabi-Yau threefolds (which are noncompact), string theorists have found an algorithm called the topological vertex [3] to compute the generating function of both open and closed Gromov-Witten invariants based on a remarkable duality with link invariants in the Chern-Simons theory approach of Witten [16, 17]. A mathematical theory of the topological vertex has been developed in [11].

The topological vertex, which is the generating function of the Gromov-Witten invariants of \mathbb{C}^3 with three special D -branes, is a mysterious combinatorial object that asks for further studies. On the A -theory side, the topological vertex can be realized as a state in the threefold tensor product of the space Λ of symmetric functions. In this representation its expressions given by physicists [3] or by mathematicians [11] are both very complicated. It is very interesting to understand the topological vertex from other perspectives. In [14], the topological vertex is related to a combinatorial problem of plane partitions. In [3] it was suggested that the topological vertex is a Bogoliubov transform (of the vacuum) via the boson-fermion correspondence. This point of view was further elaborated in [2] and extended to the partition functions of toric Calabi-Yau threefolds. Indeed, by the local mirror symmetry [8, 7], on the B-model side, one studies quantum Kodaira-Spencer theory of the local mirror curve. By physical derivations, the corresponding state is constrained by the Ward identities, giving the W_∞ constraints. (See also [5] where the partition functions are expected to be annihilated by certain quantum operators obtaining by quantizing the local mirror curves.) In this formalism it is natural to use the fermionic picture, and a simple looking formula (see §5.2) for the fermionic form of the topological vertex under the boson-fermion correspondence was conjectured in [2], which was referred to as the ADKMV conjecture in [4]. The ADKMV conjecture is directly related to integrable hierarchies: The one-legged case is related to the KP hierarchy, the two-legged case to the 2-dimensional Toda hierarchy, and the three-legged case to the 3-component KP hierarchy (see Remark 2.1). The one-legged and the two-legged cases can also be seen directly from the bosonic picture [18], but the three-legged case can only be seen through the fermionic picture.

The topological vertex can be used to compute Gromov-Witten invariants of toric Calabi-Yau 3-folds by certain gluing rules. There is a standard inner product on the space Λ of symmetric functions by setting the set of Schur functions as an orthonormal basis, and the gluing rule is essentially taking inner product over the components corresponding to the branes of gluing (see §5.1 for exact formulation). So the resulted generating functions are states in multifold tensor products of the space Λ . In general, they have very complicated combinatorial structures.

In our recent work [4], we proposed a generalization of the ADKMV conjecture to the framed topological vertex which we refer to as the framed ADKMV conjecture. Note that it is important to consider framing when we consider gluing of the topological vertex. We gave a proof in [4] of the framed ADKMV conjecture in the one-legged case and the two-legged case, and derived a determinantal formula for the framed topological vertex in the three-legged case based on the Framed ADKMV Conjecture. It remains open to give a proof of this conjecture for the full three-legged topological vertex.

Provided that the framed ADKMV conjecture holds, then a natural question is whether or not the generating functions of the Gromov-Witten invariants of general toric Calabi-Yau threefolds are Bogoliubov transforms in the fermionic picture. It was also conjectured in [2] that it is indeed the case. However, it seems very difficult to prove this conjecture directly by boson-fermion correspondence and standard Schur calculus, even for the very simple case of the resolved conifold with a single brane. In [15] the closed string partition function of the resolved conifold is related to Hall-Littlewood functions and a fermionic representation is obtained by the deformed boson-fermion correspondence. Based on the method in [2], it was shown in [10] that the B-model amplitude of the mirror space of the one-legged resolved conifold is a Bogoliubov transform, where how to use the ADKMV conjecture and the gluing rule of the topological vertex to show this result was also mentioned as an open problem. It also seems difficult to generalize the method in [2] and [10] to prove this conjecture in general.

In this paper we will tackle this problem using a different strategy. We will start from the framed ADKMV conjecture, and then consider the gluing rule of the topological vertex as presented in [3][11] in the fermionic picture. Our main aim of this article is to prove that, provided the framed ADKMV conjecture, the fermionic form of the generating function of the Gromov-Witten invariants of any toric Calabi-Yau threefold is a Bogoliubov transform (see Theorem 5.2 for exact formulation). In particular, it is a tau function of multi-component KP hierarchies. We refer to this result as the fermionic gluing principle of the topological vertex.

By the framed ADKMV conjecture for the framed one-legged and two-legged topological vertex proved in [4], we get that the generating functions of the Gromov-Witten invariants of the total spaces of the bundles $\mathcal{O}(p) \oplus \mathcal{O}(-p-2) \rightarrow \mathbb{P}^1$ with two outer branes on different vertices are two-component Bogoliubov transforms; in particular, they are tau functions of the Toda hierarchy.

In fact, we establish the gluing principle of the topological vertex by proving a gluing principle for general Bogoliubov transforms, namely, the self-gluing (see §3 for definition) of a Bogoliubov transform or the gluing (see §4.1 for definition) of two Bogoliubov transforms is still a Bogoliubov transform. It may be interesting to generalize our method to prove similar result for general tau functions of multi-component KP hierarchies which are not necessarily Bogoliubov transforms.

The rest of the paper is arranged as follows. After reviewing some preliminaries in §2, we define the self-gluing of a Bogoliubov transform and state the self-gluing principle in §3, and define the gluing of two Bogoliubov transforms and give the gluing principle in §4. In §5, we apply the results in §3 and §4 to establish the fermionic gluing principle of the topological vertex. In the final §6, we give a proof of the self-gluing principle for Bogoliubov transforms (Theorem 3.1).

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2. PRELIMINARIES

In this section, we recall briefly some well-known concepts and results that will be used later.

2.1. Partitions and symmetric functions. A partition μ of a positive integral number n is a decreasing finite sequence of integers $\mu_1 \geq \cdots \geq \mu_l > 0$, such that $|\mu| = \mu_1 + \cdots + \mu_l = n$. The following number associated to μ will be useful in this paper:

$$(1) \quad \kappa_\mu = \sum_{i=1}^l \mu_i(\mu_i - 2i + 1).$$

It is very useful to graphically represent a partition by its Young diagram. This leads to many natural definitions. First of all, by transposing the Young diagram one can define the conjugate μ^t of μ . Secondly assume the Young diagram of μ has k boxes in the diagonal. Define $m_i = \mu_i - i$ and $n_i = \mu_i^t - i$ for $i = 1, \dots, k$, then it is clear that $m_1 > \cdots > m_k \geq 0$ and $n_1 > \cdots > n_k \geq 0$. The partition μ is completely determined by the numbers m_i, n_i . We often denote the partition μ by $(m_1, \dots, m_k | n_1, \dots, n_k)$, this is called the Frobenius notation. A partition of the form $(m | n)$ in Frobenius form is called a hook partition.

Roughly speaking, a symmetric function is a symmetric polynomial of infinitely many variables (see [12] for details). We denote by Λ the space of all symmetric functions in variables $\mathbf{x} = (x_1, x_2, \dots)$. For each partition μ , there is an attached symmetric function s_μ which is called a Schur function. The Schur function corresponding to the empty partition is 1. The inner product on the space Λ is defined by setting the set of Schur functions as an orthonormal basis.

Given two partitions μ and ν , the skew Schur function $s_{\mu/\nu}$ is defined by the condition

$$(s_{\mu/\nu}, s_\lambda) = (s_\mu, s_\nu s_\lambda)$$

for all partitions λ . This is equivalent to define

$$s_{\mu/\nu} = \sum_{\lambda} c_{\nu\lambda}^\mu s_\lambda,$$

where the constants $c_{\nu\lambda}^\mu$ are the structure constants (called the Littlewood-Richardson coefficients) defined by

$$(2) \quad s_\nu s_\lambda = \sum_{\gamma} c_{\nu\lambda}^\gamma s_\gamma.$$

2.2. Fermionic Fock space. We say a set of integers $A = \{a_1, a_2, \dots\} \subset \mathbb{Z} + \frac{1}{2}$, $a_1 > a_2 > \dots$, is admissible if it satisfies the following two conditions:

1. $\mathbb{Z}_- + \frac{1}{2} \setminus A$ is finite and
2. $A \setminus \mathbb{Z}_- + \frac{1}{2}$ is finite,

where \mathbb{Z}_- is the set of negative integers.

Let W be the linear space that is spanned by the basis $\{\underline{a} | a \in \mathbb{Z} + \frac{1}{2}\}$ indexed by half-integers. For an admissible set $A = \{a_1, a_2, \dots\}$, we associate an element $\underline{A} \in \wedge^\infty W$ as follows:

$$\underline{A} = \underline{a_1} \wedge \underline{a_2} \wedge \dots.$$

Then the free fermionic Fock space \mathcal{F} is defined as

$$\mathcal{F} = \text{span}\{\underline{A} : A \subset \mathbb{Z} + \frac{1}{2} \text{ is admissible}\}.$$

We define an inner product on \mathcal{F} by taking $\{\underline{A} : A \subset \mathbb{Z} + \frac{1}{2} \text{ is admissible}\}$ as an orthonormal basis.

For $\underline{A} = \underline{a_1} \wedge \underline{a_2} \wedge \dots \in \mathcal{F}$, define its charge as:

$$|A \setminus \mathbb{Z}_- + \frac{1}{2}| - |\mathbb{Z}_- + \frac{1}{2} \setminus A|.$$

Denote by $\mathcal{F}^{(n)} \subset \mathcal{F}$ the subspace spanned by \underline{A} of charge n , then there is a decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^{(n)}.$$

An operator on \mathcal{F} is called charge 0 if it preserves the above decomposition.

The charge 0 subspace $\mathcal{F}^{(0)}$ has a basis indexed by partitions:

$$(3) \quad |\mu\rangle := \underline{\mu_1 - \frac{1}{2}} \wedge \underline{\mu_2 - \frac{3}{2}} \wedge \dots \wedge \underline{\mu_l - \frac{2l-1}{2}} \wedge \underline{-\frac{2l+1}{2}} \wedge \dots$$

where $\mu = (\mu_1, \dots, \mu_l)$, i.e., $|\mu\rangle = A_\mu$, where $A_\mu = (\mu_i - i + \frac{1}{2})_{i=1,2,\dots}$. If $\mu = (m_1, \dots, m_k | n_1, \dots, n_k)$ in Frobenius notation, then

$$(4) \quad |\mu\rangle = \underline{m_1 + \frac{1}{2}} \wedge \dots \wedge \underline{m_k + \frac{1}{2}} \wedge \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \dots \wedge \widehat{\underline{-n_k - \frac{1}{2}}} \wedge \dots \wedge \widehat{\underline{-n_1 - \frac{1}{2}}} \wedge \dots.$$

In particular, when μ is the empty partition, we get:

$$|0\rangle := \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \dots \in \mathcal{F}.$$

It will be called the fermionic vacuum vector.

We now recall the creators and annihilators on \mathcal{F} . For $r \in \mathbb{Z} + \frac{1}{2}$, define operators ψ_r and ψ_r^* by

$$\begin{aligned} \psi_r(\underline{A}) &= \begin{cases} (-1)^k \underline{a_1} \wedge \dots \wedge \underline{a_k} \wedge \underline{r} \wedge \underline{a_{k+1}} \wedge \dots, & \text{if } a_k > r > a_{k+1} \text{ for some } k, \\ 0, & \text{otherwise;} \end{cases} \\ \psi_r^*(\underline{A}) &= \begin{cases} (-1)^{k+1} \underline{a_1} \wedge \dots \wedge \widehat{\underline{a_k}} \wedge \dots, & \text{if } a_k = r \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Under the inner product defined above, for $r \in \mathbb{Z} + 1/2$, it is clear that ψ_r and ψ_r^* are adjoint operators. The anti-commutation relations for these operators are

$$(5) \quad [\psi_r, \psi_s^*]_+ := \psi_r \psi_s^* + \psi_s^* \psi_r = \delta_{r,s} id$$

and other anti-commutation relations are zero. It is clear that for $r > 0$,

$$(6) \quad \psi_{-r}|0\rangle = 0, \quad \psi_r^*|0\rangle = 0,$$

so the operators $\{\psi_{-r}, \psi_r^*\}_{r>0}$ are called the fermionic annihilators. For a partition $\mu = (m_1, m_2, \dots, m_k | n_1, n_2, \dots, n_k)$, it is clear that

$$(7) \quad |\mu\rangle = (-1)^{n_1+n_2+\dots+n_k} \prod_{i=1}^k \psi_{m_i+\frac{1}{2}} \psi_{-n_i-\frac{1}{2}}^* |0\rangle.$$

So the operators $\{\psi_r, \psi_{-r}^*\}_{r>0}$ are called the fermionic creators. The normally ordered product is defined as

$$: \psi_r \psi_r^* := \begin{cases} \psi_r \psi_r^*, & r > 0, \\ -\psi_r^* \psi_r, & r < 0. \end{cases}$$

In other words, an annihilator is always put on the right of a creator.

2.3. Boson-fermion correspondence. For any integer n , define an operator α_n on the fermionic Fock space \mathcal{F} as follows:

$$\alpha_n = \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi_r \psi_{r+n}^* :.$$

Let $\mathcal{B} = \Lambda[z, z^{-1}]$ be the bosonic Fock space, where z is a formal variable. Then the boson-fermion correspondence is a linear isomorphism $\Phi : \mathcal{F} \rightarrow \mathcal{B}$ given by

$$(8) \quad u \mapsto z^m \langle \underline{0}_m | e^{\sum_{n=1}^{\infty} \frac{p_n}{n} \alpha_n} u \rangle, \quad u \in \mathcal{F}^{(m)}$$

where $p_n = \sum_{i \geq 1} x_i^n$ are the Newton polynomials and $|\underline{0}_m\rangle = \underline{-\frac{1}{2} + m} \wedge \underline{-\frac{3}{2} + m} \wedge \dots$. It is clear that Φ induces an isomorphism between $\mathcal{F}^{(0)}$ and Λ . Explicitly, this isomorphism is given by

$$(9) \quad |\mu\rangle \longleftrightarrow s_\mu.$$

The boson-fermionic correspondence plays an important role in Kyoto school's theory of integrable hierarchies. For example,

Proposition 2.1. *If $\tau \in \Lambda$ corresponds to $|v\rangle \in F^{(0)}$, then τ is a tau-function of the KP hierarchy in the Miwa variable $t_n = \frac{p_n}{n}$ if and only if $|v\rangle$ satisfies the bilinear relation*

$$(10) \quad \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r |v\rangle \otimes \psi_r^* |v\rangle = 0.$$

Remark 2.1. A state $|v\rangle \in \mathcal{F}^{(0)}$ satisfies the bilinear relation (10) if and only if it lies in the orbit $\widehat{GL}_\infty |0\rangle$. This is equivalent to say that $|v\rangle$ can be represented as

$$|v\rangle = \exp\left(\sum_{r,s \in \mathbb{Z} + 1/2} M_{rs} : \psi_r \psi_s^* : \right) |0\rangle$$

for some coefficients M_{rs} . There is also a multi-component generalization of the boson-fermion correspondence which can be used to study multi-component KP hierarchies [9].

3. SELF-GLUING PRINCIPLE FOR BOGOLIUBOV TRANSFORMS

In this section, we introduce the notion of Bogoliubov transforms and their self-gluing, and give a statement of the self-gluing principle of Bogoliubov transforms.

3.1. Bogoliubov transforms. On the N -component fermionic Fock space $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N$, where $\mathcal{F}_1, \dots, \mathcal{F}_N$ are N -copies of \mathcal{F} , define operators ψ_r^i and ψ_r^{i*} , for $r \in \mathbb{Z} + \frac{1}{2}$ and $i = 1, \dots, N$. They act on the i -th factor of the tensor product as the operators ψ_r and ψ_r^* respectively, and we use the Koszul sign convention for the anti-commutation relations for these operators, i.e., we set

$$(11) \quad [\psi_r^i, \psi_s^j]_+ = [\psi_r^i, \psi_s^{j*}]_+ = [\psi_r^{i*}, \psi_s^{j*}]_+ = 0$$

for $i \neq j$ and $r, s \in \mathbb{Z} + \frac{1}{2}$.

For an integer n , we denote by \mathbf{n} the number $n + 1/2$.

We call a vector $V \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N$ a Bogoliubov transform (of the vacuum) if it is gotten from the vacuum in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N$ acted upon by an exponential of a quadratic expression of fermionic creators. In other word, it can be represented as

$$(12) \quad V = \exp\left(\sum_{i,j=1}^N \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0\rangle,$$

where A_{mn}^{ij} are certain coefficients possibly with parameters. Here and in the following, if not specified otherwise, for simplicity of notations we will use $|0\rangle$ to denote the vacuum $|0\rangle_1 \otimes \cdots \otimes |0\rangle_N$ in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N$ and similar tensor products, the exact meaning will be clear from the context.

One can see directly from the definition that Bogoliubov transforms are tau functions of multi-component KP hierarchies constructed in [9].

3.2. The gluing vectors. Let a and b be two indices, \mathcal{F}_a and \mathcal{F}_b two copies of the fermionic Fock space \mathcal{F} . We call a vector $P_{ab}^{\mathbf{E}} \in \mathcal{F}_a \otimes \mathcal{F}_b$ of the form

$$(13) \quad P_{ab}^{\mathbf{E}} = \exp\left(\sum_{i,j=a,b} \sum_{m,n \geq 0} Q^{m+n+1} \Theta^{\epsilon_{ij}^{ab}} E_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0\rangle$$

a *gluing vector*, where Q , Θ are formal variables and $\mathbf{E} = \{E_{mn}^{ij} | m, n \geq 0; i, j = a, b\}$ is a series of coefficients maybe with parameters, and ϵ_{ij}^{ab} is given by

$$\epsilon_{ij}^{ab} = \begin{cases} 1, & \text{if } i = a, j = b, \\ -1, & \text{if } i = b, j = a, \\ 0, & \text{if } i = j = a \text{ or } b. \end{cases}$$

Here $P_{ab}^{\mathbf{E}}$ is viewed as a formal power series of Q , Θ and Θ^{-1} with coefficients in $\mathcal{F}_a \otimes \mathcal{F}_b$; it can also be viewed as a vector in the two-component fermionic Fock space $\mathcal{F}_a \otimes \mathcal{F}_b$ with certain parameters.

When $P_{ab}^{\mathbf{E}}$ is viewed as a Laurent series in Θ , write

$$P_{ab}^{\mathbf{E}} = \sum_{n \in \mathbb{Z}} P_n \Theta^n.$$

Then it is easy to see that

$$P_n \in \mathcal{F}_a^{(n)} \otimes \mathcal{F}_b^{(-n)}.$$

In particular,

$$P_0 \in \mathcal{F}_a^{(0)} \otimes \mathcal{F}_b^{(0)}.$$

3.3. Self-gluings of Bogoliubov transforms. Let V be a Bogoliubov transform in the $(M+2)$ -component fermionic Fock space $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a \otimes \mathcal{F}_b$. We write

$$V = \exp \left(\sum_{i,j \in \{a,b,1,2,\dots,M\}} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) |0\rangle,$$

where A_{mn}^{ij} are certain coefficients maybe with parameters. There is a natural inner product on $\mathcal{F}_a \otimes \mathcal{F}_b$ which is induced from that on \mathcal{F} . The *self-gluings* $\tilde{G}^{\mathbf{E}}(V)$ of V with the gluing vector $P_{ab}^{\mathbf{E}}$ is defined to be the inner product

$$(14) \quad \tilde{G}^{\mathbf{E}}(V) = (V, P_{ab}^{\mathbf{E}}) \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M,$$

where the inner product is taken over the components $\mathcal{F}_a \otimes \mathcal{F}_b$. We view $\tilde{G}^{\mathbf{E}}(V)$ as a formal power series of Q , Θ and Θ^{-1} with coefficients in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$. The closed part $\tilde{G}^{\mathbf{E}}(V)_{closed}$ of the self-gluings is defined to be the inner product

$$(15) \quad \tilde{G}^{\mathbf{E}}(V)_{closed} = (\exp \left(\sum_{i,j=a,b} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) |0\rangle_{ab}, P_{ab}^{\mathbf{E}}).$$

In the above expressions, $|0\rangle_{ab}$ denotes the vacuum in $\mathcal{F}_a \otimes \mathcal{F}_b$. The quotient

$$(16) \quad G^{\mathbf{E}}(V) := \tilde{G}^{\mathbf{E}}(V) / \tilde{G}^{\mathbf{E}}(V)_{closed}$$

is called the *normalized self-gluings* $G^{\mathbf{E}}(V)$ of V with the gluing vector $P_{ab}^{\mathbf{E}}$. We view $G^{\mathbf{E}}(V)$ as a vector in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$ with parameters, as well as a formal power series of Q , Θ and Θ^{-1} with coefficients in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$.

3.4. The self-gluings principle for Bogoliubov transforms. One of the main aims of this paper is to prove that the normalized self-gluings of an arbitrary Bogoliubov transform is again a Bogoliubov transform. In particular, it is a tau function of multi-component KP hierarchies. We refer to this result as the self-gluings principle for Bogoliubov transforms.

Theorem 3.1. *Let V be a Bogoliubov transform in the $(M+2)$ -component fermionic Fock space $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a \otimes \mathcal{F}_b$, $M > 0$. Then the normalized self-gluings $G^{\mathbf{E}}(V)$ of V defined as in §3.3 is again a Bogoliubov transform in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$.*

The proof of Theorem 3.1 will be presented in §6.

4. GLUING OF TWO BOGOLIUBOV TRANSFORMS

In this section, we apply the result in §3 to study the gluing of two arbitrary Bogoliubov transforms.

4.1. Gluing principle for Bogoliubov transforms. Let $V_1 \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a$ and $V_2 \in \mathcal{F}_{M+1} \otimes \cdots \otimes \mathcal{F}_{M+N} \otimes \mathcal{F}_b$ be two Bogoliubov transforms. We write

$$V_1 = \exp \left(\sum_{i,j \in \{1,\dots,M,a\}} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) |0\rangle,$$

$$V_2 = \exp \left(\sum_{i,j \in \{b,M+1,\dots,M+N\}} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) |0\rangle,$$

where A_{mn}^{ij} are certain coefficients maybe with parameters. Then their tensor product $V_1 \otimes V_2$ is a Bogoliubov transform in the $(M+N+2)$ -component fermionic Fock space $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_{M+N} \otimes \mathcal{F}_a \otimes \mathcal{F}_b$. It has the form

$$(17) \quad V_1 \otimes V_2 = \exp \left(\sum_{i,j \in \{a,b,1,\dots,M+N\}} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) |0\rangle,$$

where we set $A_{mn}^{ij} = 0$ if i and j are not in the same set $\{a, 1, \dots, M\}$ or $\{b, M+1, \dots, M+N\}$. With the gluing vector $P_{ab}^{\mathbf{E}}$ defined in (13), we define the gluing $\tilde{G}^{\mathbf{E}}(V_1, V_2)$, the closed part of the gluing $\tilde{G}^{\mathbf{E}}(V_1, V_2)_{\text{closed}}$, and the normalized gluing $G^{\mathbf{E}}(V_1, V_2)$ of V_1 and V_2 to be $\tilde{G}^{\mathbf{E}}(V_1 \otimes V_2)$, $\tilde{G}^{\mathbf{E}}(V_1 \otimes V_2)_{\text{closed}}$, and $G^{\mathbf{E}}(V_1 \otimes V_2)$ respectively. Note that we have $G^{\mathbf{E}}(V_1, V_2) = \tilde{G}^{\mathbf{E}}(V_1, V_2) / \tilde{G}^{\mathbf{E}}(V_1, V_2)_{\text{closed}}$.

By the self-gluing principle for Bogoliubov transforms stated in Theorem 3.1, we immediately get the gluing principle for two Bogoliubov transforms, which says that the normalized gluing of two Bogoliubov transforms is again a Bogoliubov transform.

Theorem 4.1. *Let $V_1 \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a$ and $V_2 \in \mathcal{F}_{M+1} \otimes \cdots \otimes \mathcal{F}_{M+N} \otimes \mathcal{F}_b$ be two Bogoliubov transforms. Then the normalized gluing $G^{\mathbf{E}}(V_1, V_2)$ of V_1 and V_2 is a Bogoliubov transform in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_{M+N}$.*

Remark 4.1. It may be interesting to consider the gluing of two general tau functions of multi-component KP hierarchies. Let $V_1 \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a$ and $V_2 \in \mathcal{F}_{M+1} \otimes \cdots \otimes \mathcal{F}_N \otimes \mathcal{F}_b$ that can be represented as

$$V_1 = \exp \left(\sum_{i,j \in \{a,1,\dots,M\}} \sum_{m,n \in \mathbb{Z}} A_{mn}^{ij} : \psi_{\mathbf{m}}^i \psi_{\mathbf{n}}^{j*} : \right) |0\rangle,$$

$$V_2 = \exp \left(\sum_{i,j \in \{b,M+1,\dots,M+N\}} \sum_{m,n \in \mathbb{Z}} A_{mn}^{ij} : \psi_{\mathbf{m}}^i \psi_{\mathbf{n}}^{j*} : \right) |0\rangle.$$

With the gluing vectors $P_{ab}^{\mathbf{E}}$, we can define the gluing and normalized gluing of V_1 and V_2 similarly as in the case of Bogoliubov transforms. Motivated by Theorem 4.1, it is natural to expect that the normalized gluing of V_1 and V_2 is again a tau function of the $(M+N)$ -component KP hierarchy. The rough meaning here is that the gluing of two KP integrable systems may give us a new KP integrable system.

4.2. Specialization of the gluing vectors. In the context of topological vertex theory, we need to consider a special kind of gluing of fermionic states. As we will see, this can be realized as the gluing of Bogoliubov transforms defined in the above subsections, with a special choice of the gluing vectors.

The gluing vectors we need, denoted by $|P_{ab}^f\rangle$, are vectors in $\mathcal{F}_a \otimes \mathcal{F}_b$ depending on an integer f represented as

$$(18) \quad |P_{ab}^f\rangle = \exp \left[\sum_{n=0}^{\infty} Q^{n+1/2} (\Theta \epsilon_n \psi_{\mathbf{n}}^a \psi_{-\mathbf{n}}^{b*} + \Theta^{-1} \epsilon'_n \psi_{\mathbf{n}}^b \psi_{-\mathbf{n}}^{a*}) \right] |0\rangle,$$

where Q and Θ are formal variables, ϵ_n, ϵ'_n are coefficients with a parameter q that are given by

$$(19) \quad \begin{aligned} \epsilon_n &= i^{1+f} (-1)^{(1+f)n} q^{fn(n+1)/2}, \\ \epsilon'_n &= i^{1+f} (-1)^{(1+f)n} q^{-fn(n+1)/2}. \end{aligned}$$

The vectors $|P_{ab}^f\rangle$ were introduced in [2] with the motivation of the gluing rule for the topological vertex. They can be viewed as a formal power series of $Q^{1/2}$, Θ and Θ^{-1} with coefficients in $\mathcal{F}_a \otimes \mathcal{F}_b$ with parameters. If we replace Q by $Q^{1/2}$ and set

$$E_{mn}^{ij} = \begin{cases} \delta_{mn} \epsilon_n, & \text{if } i = a, j = b, \\ \delta_{mn} \epsilon'_n, & \text{if } i = b, j = a, \end{cases}$$

in (13), then we get $|P_{ab}^f\rangle = P_{ab}^{\mathbf{E}}$.

When considering the self-gluing and gluing of Bogoliubov transforms with respect to the gluing vector $|P_{ab}^f\rangle$, we will denote $\tilde{G}^{\mathbf{E}}(V_1, V_2)$, $\tilde{G}^{\mathbf{E}}(V_1, V_2)_{\text{closed}}$, $G^{\mathbf{E}}(V_1, V_2)$ by $\tilde{G}^f(V_1, V_2)$, $\tilde{G}^f(V_1, V_2)_{\text{closed}}$, $G^f(V_1, V_2)$ respectively, and $\tilde{G}^{\mathbf{E}}(V)$, $\tilde{G}^{\mathbf{E}}(V)_{\text{closed}}$, $G^{\mathbf{E}}(V)$ by $\tilde{G}^f(V)$, $\tilde{G}^f(V)_{\text{closed}}$, $G^f(V)$ respectively.

A direct corollary of Theorem 3.1 and Theorem 4.1 is the following

Theorem 4.2. 1). Let $V \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a \otimes \mathcal{F}_b$ be a Bogoliubov transform. Then the normalized self-gluing $G^f(V)$ is a Bogoliubov transform in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$; 2). Let $V_1 \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M \otimes \mathcal{F}_a$ and $V_2 \in \mathcal{F}_{M+1} \otimes \cdots \otimes \mathcal{F}_{M+N} \otimes \mathcal{F}_b$ be two Bogoliubov transforms. Then the normalized gluing $G^f(V_1, V_2)$ is a Bogoliubov transform in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_{M+N}$.

The following lemma, which was also observed in [2], implies that the gluing of Bogoliubov transforms with $|P_{ab}^f\rangle$ as gluing vectors indeed coincides with the gluing of topological vertex given in [3] and [11].

Lemma 4.3. Let μ_a and μ_b be two partitions, then

$$(20) \quad \langle \mu_a | \otimes \langle \mu_b | P_{ab}^f \rangle = \delta_{\mu_a \mu_b} (-1)^{(f+1)|\mu_a|} q^{f\kappa_{\mu_a}/2} Q^{|\mu|}.$$

Proof. Let $A_n = Q^{n+1/2} \Theta \epsilon_n$, $B_n = Q^{n+1/2} \Theta^{-1} \epsilon'_n$. By (5), we have

$$|P_{ab}^f\rangle = \prod_{n=1}^{\infty} (1 + A_n \psi_{\mathbf{n}}^a \psi_{-\mathbf{n}}^{b*}) \prod_{n=1}^{\infty} (1 + B_n \psi_{\mathbf{n}}^b \psi_{-\mathbf{n}}^{a*}).$$

If we denote by $|P_{ab}^f\rangle_0$ the projection of $|P_{ab}^f\rangle$ on the subspace $\mathcal{F}_a^{(0)} \otimes \mathcal{F}_b^{(0)}$ with respect to the decomposition

$$(\mathcal{F}_a \otimes \mathcal{F}_b)^{(0)} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{F}_a^{(n)} \otimes \mathcal{F}_b^{(-n)}),$$

then we have

$$\begin{aligned}
|P_{ab}^f\rangle_0 &= \sum_{\substack{m_1 > \dots > m_k \geq 0 \\ n_1 > \dots > n_k \geq 0}} \prod_{i=1}^k A_{m_i} B_{n_i} \psi_{\mathbf{m}_i}^a \psi_{-\mathbf{m}_i}^{b*} \psi_{\mathbf{n}_i}^b \psi_{-\mathbf{n}_i}^{a*} |0\rangle \\
&= \sum_{\gamma=(m_1, \dots, m_k | n_1, \dots, n_k)} \prod_{i=1}^k (-1)^{m_i+n_i} A_{m_i} B_{n_i} |\gamma\rangle \otimes |\gamma^t\rangle \\
&= \sum_{\gamma} \prod_{i=1}^k Q^{m_i+n_i+1} (-1)^{(1+f)(m_i+n_i+1)} q^{f(m_i(m_i+1)-n_i(n_i+1))/2} |\gamma\rangle \otimes |\gamma^t\rangle.
\end{aligned}$$

By lemma ??, we see

$$|P_{ab}^f\rangle_0 = \sum_{\gamma} (-1)^{(1+f)|\gamma|} q^{f\kappa_{\gamma}/2} Q^{|\gamma|} |\gamma\rangle \otimes |\gamma^t\rangle.$$

By the definition of the inner product on $\mathcal{F}_a \otimes \mathcal{F}_b$, we have

$$\langle \mu_a | \otimes \langle \mu_b | P_{ab}^f \rangle = \delta_{\mu_a \mu_b^t} (-1)^{(f+1)|\mu_a|} q^{f\kappa_{\mu_a}/2} Q^{|\mu_a|}.$$

□

5. FERMIONIC GLUING PRINCIPLE OF THE TOPOLOGICAL VERTEX

In this section, we apply the results in previous sections to study gluing of the topological vertex in the fermionic picture. The main aim here is to establish the fermionic gluing principle of the topological vertex, that is, assuming the framed ADKMV conjecture, the generating functions of the open Gromov-Witten invariants of all toric Calabi-Yau threefolds are Bogoliubov transforms (see Theorem 5.2 for the exact formulation).

5.1. The topological vertex and its gluing. We give a brief introduction to the theory of topological vertex that introduced in [3], with a formulation that fits for our context.

To each toric Calabi-Yau threefold X , we can attach a trivalent planar graph Γ to it that encodes the loci of degeneration of the $T^2 \times \mathbb{R}$ fibration of X over \mathbb{R}^3 . The planar graph Γ is called the *toric diagram* of X (see [3] for details). The toric diagram of \mathbb{C}^3 is a trivalent vertex. Each toric Calabi-Yau threefold can be constructed by gluing \mathbb{C}^3 pieces, which reflects the fact that each toric diagram can be constructed by gluing trivalent vertices.

The generating function of Gromov-Witten invariants of \mathbb{C}^3 is given by the topological vertex introduced in [3] and [11]. The topological vertex introduced in [3] is defined by

$$(21) \quad W_{\mu^1, \mu^2, \mu^3}(q) = \sum_{\rho^1, \rho^3} c_{\rho^1(\rho^3)^t}^{\mu^1(\mu^3)^t} q^{\kappa_{\mu^2/2} + \kappa_{\mu^3/2}} \frac{W_{(\mu^2)^t \rho^1}(q) W_{\mu^2(\rho^3)^t}(q)}{W_{\mu^2 \emptyset}(q)},$$

where

$$c_{\rho^1(\rho^3)^t}^{\mu^1(\mu^3)^t} = \sum_{\eta} c_{\eta \rho^1}^{\mu^1} c_{\eta(\rho^3)^t}^{(\mu^3)^t},$$

and the constants $c_{\nu\lambda}^\mu$ are the Littlewood-Richardson coefficients defined in §2.1. It can also be represented in terms of specialization of skew Schur functions as follows (see e.g. [19]):

$$(22) \quad W_{\mu^1, \mu^2, \mu^3}(q) = (-1)^{|\mu^2|} q^{\kappa_{\mu^3}/2} s_{(\mu^2)^t}(q^{-\rho}) \sum_{\eta} s_{\mu^1/\eta}(q^{(\mu^2)^t + \rho}) s_{(\mu^3)^t/\eta}(q^{\mu^2 + \rho}),$$

where $q^{\mu+\rho} = (q^{\mu_i - i + 1/2})_{i=1,2,\dots}$.

It is also necessary to consider framings of \mathbb{C}^3 and their effects on the Gromov-Witten invariants. For the special case \mathbb{C}^3 , if we index the three edges of the trivalent vertex by 1, 2 and 3 clockwise, then the framing of \mathbb{C}^3 can be labeled by elements in \mathbb{Z}^3 . The topological vertex given by (21) encodes the Gromov-Witten invariants of \mathbb{C}^3 with the canonical framing $(0, 0, 0)$. In general, the framed topological vertex with framing (a_1, a_2, a_3) , which encodes the Gromov-Witten invariants of \mathbb{C}^3 with framing (a_1, a_2, a_3) , is given by:

$$(23) \quad W_{\mu^1, \mu^2, \mu^3}^{(a_1, a_2, a_3)}(q) = (-1)^{\sum_{i=1}^3 |\mu^i| a_i} q^{\sum_{i=1}^3 a_i \kappa_{\mu^i}/2} W_{\mu^1, \mu^2, \mu^3}(q).$$

The corresponding generating function

$$(24) \quad Z^{(a_1, a_2, a_3)}(q; \mathbf{x}^1; \mathbf{x}^2; \mathbf{x}^3) = \sum_{\mu^1, \mu^2, \mu^3} W_{\mu^1, \mu^2, \mu^3}^{(a_1, a_2, a_3)}(q) s_{\mu^1}(\mathbf{x}^1) s_{\mu^2}(\mathbf{x}^2) s_{\mu^3}(\mathbf{x}^3),$$

is a vector in $\Lambda^{\otimes 3}$, where, as before, Λ is the space of symmetric functions.

Now we consider general toric Calabi-Yau threefolds. Let X be a toric Calabi-Yau threefold whose toric diagram is Γ . Assume Γ has h vertices and g loops. Then the Gromov-Witten invariants of X with arbitrary framing can be deduced inductively from the framed topological vertex by gluing procedure as follows:

Case 1. Assume Γ can be constructed by gluing another toric diagram Γ' and the trivalent vertex along a noncompact edge. Then the number of vertices of Γ' is $h-1$. Denote by L_1, \dots, L_{n-2}, L_a the noncompact edges of Γ' and by L_b, L_{n-1}, L_n the edges of the trivalent vertex ordered clockwise. Assume we glue Γ' and the trivalent vertex along L_a and L_b . Assume the generating function of the Gromov-Witten invariants of the toric Calabi-Yau threefold corresponding to Γ' with framing a_1, \dots, a_{n-2} on edges L_1, \dots, L_{n-2} and with canonical framing on the edge L_a is given by

$$Z'(\mathbf{x}^1, \dots, \mathbf{x}^{n-2}, \mathbf{x}^a) = \sum_{\mu^1, \dots, \mu^{n-2}, \mu^a} Z'_{\mu^1, \dots, \mu^{n-2}, \mu^a} s_{\mu^1}(\mathbf{x}^1) \cdots s_{\mu^{n-2}}(\mathbf{x}^{n-2}) s_{\mu^a}(\mathbf{x}^a),$$

which is viewed as a vector in $\Lambda^{\otimes (n-1)}$ with certain parameters. Then the generating function Z of the Gromov-Witten invariants of X with framing a_1, \dots, a_n on the edges L_1, \dots, L_n is given by

$$(25) \quad \begin{aligned} & Z(\mathbf{x}^1, \dots, \mathbf{x}^n) \\ &= \sum_{\mu^1, \dots, \mu^n, \mu} Z'_{\mu^1, \dots, \mu^{n-2}, \mu} (-1)^{|\mu|} Q^{|\mu|} W_{\mu^t, \mu^{n-1}, \mu^n}^{(f, a_{n-1}, a_n)} s_{\mu^1}(\mathbf{x}^1) \cdots s_{\mu^n}(\mathbf{x}^n) \\ &= \sum_{\mu^1, \dots, \mu^n, \mu} Z'_{\mu^1, \dots, \mu^{n-2}, \mu} (-1)^{(f+1)|\mu|} q^{\kappa_{\mu}/2} Q^{|\mu|} W_{\mu^t, \mu^{n-1}, \mu^n}^{(0, a_{n-1}, a_n)} s_{\mu^1}(\mathbf{x}^1) \cdots s_{\mu^n}(\mathbf{x}^n), \end{aligned}$$

where $Q = e^{-t}$ and t is the Kähler parameter corresponding to the compact edge obtained by gluing L_a and L_b . The framing f on the edge L_b in (25) is determined

by the compatibility with the canonical framing on the edge L_a (for more details, see [3]).

Case 2. Assume Γ can be constructed from another toric diagram Γ'' by gluing two noncompact edges. Then the number of loops in Γ'' is $g-1$. Let $L_1, \dots, L_n, L_a, L_b$ be noncompact edges of Γ'' , and assume Γ is obtained from Γ'' by gluing the edges L_a and L_b . Assume

$$\begin{aligned} & Z''(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{x}^a, \mathbf{x}^b) \\ = & \sum_{\mu^1, \dots, \mu^n, \mu_a, \mu_b} Z''_{\mu^1, \dots, \mu^n, \mu_a, \mu_b} s_{\mu^1}(\mathbf{x}^1) \cdots s_{\mu^n}(\mathbf{x}^n) s_{\mu^a}(\mathbf{x}^a) s_{\mu^b}(\mathbf{x}^b) \end{aligned}$$

is the generating function of the Gromov-Witten invariants of the toric Calabi-Yau threefold X'' corresponding to Γ'' with framing a_1, \dots, a_n on the edges L_1, \dots, L_n and canonical framing on the two edges L_a and L_b . Then the generating function Z of the Gromov-Witten invariants of X with framing a_1, \dots, a_n on the edges L_1, \dots, L_n is given by

$$(26) \quad \begin{aligned} & Z(\mathbf{x}^1, \dots, \mathbf{x}^n) \\ = & \sum_{\mu^1, \dots, \mu^n, \mu} Z''_{\mu^1, \dots, \mu^n, \mu, \mu^t} (-1)^{(f+1)|\mu|} Q^{|\mu|} q^{-f\kappa_u/2} s_{\mu^1}(\mathbf{x}^1) \cdots s_{\mu^n}(\mathbf{x}^n), \end{aligned}$$

where $Q = e^{-t}$ as above and t is the Kähler parameter corresponding to the compact edge obtained by gluing L_a and L_b , and we also require that the framing f on the edge L_b matches with the canonical framing on L_a .

In conclusion, we can compute by induction (with respect to the numbers of vertices and loops of the toric diagrams) the Gromov-Witten invariants of all toric Calabi-Yau threefolds with arbitrary framing, with the framed topological vertex as the initial datum. In this process, two Kähler parameters corresponding to two compact edges are identified if the two edges give rise to two homologous copies of \mathbb{P}^1 in X .

5.2. Fermionic representation of the framed topological vertex. The framed topological vertex given in (24) is a vector in the space $\Lambda^{\otimes 3}$, where Λ is the space of symmetric functions. By the boson-fermion correspondence, it corresponds to a vector in $\mathcal{F}_1^{(0)} \otimes \mathcal{F}_2^{(0)} \otimes \mathcal{F}_3^{(0)} \subset \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$, where $\mathcal{F}_i, i = 1, 2, 3$ are three copies of the fermionic Fock space \mathcal{F} , and $\mathcal{F}_i^{(0)}$ is the charge 0 subspace of \mathcal{F}_i . Though the topological vertex given in (21) has an extremely complicated expression in terms of symmetric functions, it is conjectured in [3] and [2] that it has a simple expression in the fermionic picture. This conjecture was referred to as the ADKMV conjecture in [4]. The ADKMV Conjecture states that, for arbitrary partitions μ^1, μ^2, μ^3 ,

$$(27) \quad W_{\mu^1, \mu^2, \mu^3}(q) = \langle \mu^1 | \otimes \langle \mu^2 | \otimes \langle \mu^3 | \exp \left(\sum_{i,j=1}^3 \sum_{m,n \geq 0} A_{mn}^{ij}(q) \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) | 0 \rangle \otimes | 0 \rangle \otimes | 0 \rangle,$$

where for $i = 1, 2, 3$, the coefficients are given by

$$\begin{aligned} A_{mn}^{ii}(q) &= (-1)^n \frac{q^{m(m+1)/4 - n(n+1)/4}}{[m+n+1][m]![n]!}, \\ A_{mn}^{i(i+1)}(q) &= (-1)^n q^{m(m+1)/4 - n(n+1)/4 + 1/6} \sum_{l=0}^{\min(m,n)} \frac{q^{(l+1)(m+n-l)/2}}{[m-l]![n-l]!}, \\ A_{mn}^{i(i-1)}(q) &= (-1)^{n+1} q^{-m(m+1)/4 + n(n+1)/4 - 1/6} \sum_{l=0}^{\min(m,n)} \frac{q^{-(l+1)(m+n-l)/2}}{[m-l]![n-l]!}. \end{aligned}$$

Here it is understood that $A^{34} = A^{30}$ and $A^{10} = A^{13}$.

The ADKMV conjecture was generalized to the framed topological vertex in [4] as:

$$(28) \quad W_{\mu^1, \mu^2, \mu^3}^{(\mathbf{a})}(q) = \langle \mu^1 | \otimes \langle \mu^2 | \otimes \langle \mu^3 | \exp \left(\sum_{i,j=1}^3 \sum_{m,n \geq 0} A_{mn}^{ij}(q; \mathbf{a}) \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) | 0 \rangle \otimes | 0 \rangle \otimes | 0 \rangle$$

for $A_{mn}^{ij}(q; \mathbf{a})$ similar to $A_{mn}^{ij}(q)$ above:

$$\begin{aligned} A_{mn}^{ii}(q; \mathbf{a}) &= (-1)^{(m+n+1)a_i + n} q^{(2a_i+1)(m(m+1)-n(n+1))/4} \frac{1}{[m+n+1][m]![n]!}, \\ A_{mn}^{i(i+1)}(q; \mathbf{a}) &= (-1)^{ma_i + (n+1)a_{i+1} + n} q^{\frac{(2a_i+1)m(m+1) - (2a_{i+1}+1)n(n+1)}{4} + \frac{1}{6}} \\ &\quad \sum_{l=0}^{\min(m,n)} \frac{q^{\frac{1}{2}(l+1)(m+n-l)}}{[m-l]![n-l]!}, \\ A_{mn}^{i(i-1)}(q; \mathbf{a}) &= -(-1)^{ma_i + (n+1)a_{i-1} + n} q^{\frac{(2a_i+1)m(m+1) - (2a_{i-1}+1)n(n+1)}{4} - \frac{1}{6}} \\ &\quad \sum_{l=0}^{\min(m,n)} \frac{q^{-\frac{1}{2}(l+1)(m+n-l)}}{[m-l]![n-l]!}. \end{aligned}$$

Here $\mathbf{a} = a_1, a_2, a_3$. We refer to this conjecture as the Framed ADKMV Conjecture. The Framed ADKMV Conjecture for the one-legged and two-legged framed topological vertex was proved in [4]. It remains open to give a proof of the (framed) ADKMV conjecture for the full three-legged topological vertex.

A straightforward application of the ADKMV Conjecture and the Framed ADKMV Conjecture is that they establish a connection between the topological vertex and integrable hierarchies as pointed out in [2].

5.3. Fermionic gluing principle of the topological vertex - simple cases.

Let $V_0^{(a_1, a_2, a_3)} \in \mathcal{F}_1^{(0)} \otimes \mathcal{F}_2^{(0)} \otimes \mathcal{F}_3^{(0)}$ be the vector corresponding to the framed topological vertex given by (24) under the boson-fermion correspondence. The framed ADKMV conjecture stated in the above subsection implies that there is a Bogoliubov transform $V^{(a_1, a_2, a_3)} \in \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$ such that its projection to the subspace $\mathcal{F}_1^{(0)} \otimes \mathcal{F}_2^{(0)} \otimes \mathcal{F}_3^{(0)}$ with respect to the charge decomposition coincides with $V_0^{(a_1, a_2, a_3)}$.

To show clearly how the fermionic gluing principle works, we first consider two simple examples, with the framed ADKMV conjecture as the starting point.

The first example is the total space X_p of the bundle $\mathcal{O}(p) \oplus \mathcal{O}(-p-2) \rightarrow \mathbb{P}^1$, $p \in \mathbb{Z}$. The toric diagram Γ_{X_p} of X_p contains two vertices, which reflects that X_p can be constructed by gluing two \mathbb{C}^3 pieces. Let L_1, \dots, L_4 be the noncompact edges of Γ_{X_p} , with L_1, L_2 in one vertex and L_3, L_4 in the other. Let \tilde{Z} be the generating function of the Gromov-Witten invariants of X_p , and let Z_{closed} be the generating function of the closed Gromov-Witten invariants (i.e., the Gromov-Witten invariants with no branes) of X_p . Then $Z_{X_p} := \tilde{Z}/Z_{closed}$ is the generating function of the open Gromov-Witten invariants of X_p . Under the boson-fermion correspondence (a multi-component version of (9)), Z_{X_p} corresponds to a vector, say V_{X_p} , in $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_4^{(0)}$.

For an integer f , provided the framed ADKMV conjecture, we can define the normalized gluing $V = G^f(V^{(a_1, a_2, 0)}, V^{(0, a_3, a_4)})$ of $V^{(a_1, a_2, 0)}$ and $V^{(0, a_3, a_4)}$ with the special gluing vector $|P_{ab}^f\rangle$ defined as in §4.2, where we label the edges of the first vertex by L_1, L_2, L_a , and label those of the second vertex by L_b, L_3, L_4 . By Theorem 4.2, V is a Bogoliubov transform in $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_4$. We assume Γ_{X_p} is constructed by gluing the two vertices along L_a and L_b . By Lemma 4.3 and the gluing rule of the topological vertex given in (25), for a suitable choice of f (depends on p), the projection of V on $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_4^{(0)}$ coincides with V_{X_p} . In other words, if we write

$$Z_{X_p}(\mathbf{x}^1, \dots, \mathbf{x}^4) = \sum_{\mu_1, \dots, \mu_4} Z_{\mu_1 \dots \mu_4} s_{\mu_1}(\mathbf{x}^1) \dots s_{\mu_4}(\mathbf{x}^4),$$

then we have $Z_{\mu_1 \dots \mu_4} = \langle \mu_1 | \otimes \dots \otimes \langle \mu_4 | V \rangle$, for all partitions μ_1, \dots, μ_4 .

Combine Theorem 5.2 and the solution in [4] of the famed ADKMV conjecture for the cases of the one-legged and the two-legged framed topological vertex, we directly get

Theorem 5.1. *Let X_p be the total space of the bundle $\mathcal{O}(p) \otimes \mathcal{O}(-p-2) \rightarrow \mathbb{P}^1$, $p \in \mathbb{Z}$. Let*

$$Z_{X_p}(\mathbf{x}, \mathbf{y}) = \sum_{\mu, \nu} Z_{\mu, \nu} s_{\mu}(\mathbf{x}) s_{\nu}(\mathbf{y})$$

be the generating function of the open Gromov-Witten invariants of X with two outer branes on different vertices. Then there is a Bogoliubov transform $V \in \mathcal{F}_1 \otimes \mathcal{F}_2$ such that

$$Z_{\mu, \nu} = \langle \mu | \otimes \langle \nu | V \rangle$$

for all partitions μ, ν .

The second example is the local \mathbb{P}^2 , i.e., the total space of the vector bundle $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$. We denote it by X . Its toric diagram Γ_X contains three vertices and a loop. So Γ_X can be constructed by gluing two noncompact edges of a toric diagram, say Γ' , with three vertices and no loops. Let X' be the toric Calabi-Yau threefold corresponding to Γ' . Repeat the process as above, one can show that there is a Bogoliubov transform $V' \in \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \otimes \mathcal{F}_a \otimes \mathcal{F}_b$ whose projection on $\mathcal{F}_1^{(0)} \otimes \mathcal{F}_2^{(0)} \otimes \mathcal{F}_3^{(0)} \otimes \mathcal{F}_a^{(0)} \otimes \mathcal{F}_b^{(0)}$ corresponds, under boson-fermion correspondence, to the generating function of the open Gromov-Witten invariants of X' with framings. Here we label the noncompact edges of Γ' by L_1, L_2, L_3, L_a and L_b such that Γ_X is constructed from Γ' by gluing L_a and L_b . Provided the framed ADKMV conjecture, we can define the normalized self-gluing $G^f(V')$ of V' with the special gluing vector $|P_{ab}^f\rangle$. By Theorem 4.2, $G^f(V')$ is a Bogoliubov transform in $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$. Unlike

the situation in the first example, the projection of V on $\mathcal{F}_1^{(0)} \otimes \mathcal{F}_2^{(0)} \otimes \mathcal{F}_3^{(0)}$, denoted by V_0 , contains a new formal variable Θ that doesn't appear in Z_X , the generating function of the open Gromov-Witten invariants of X . On the other hand, also by Lemma 4.3 and the gluing rule of the topological vertex given in (26), we see that the constant term (with respect to Θ) corresponds to Z_X under the boson-fermion correspondence. In other words, for any partitions μ_i , $i = 1, 2, 3$, the open Gromov-Witten invariants Z_{μ_1, μ_2, μ_3} of X with boundary conditions given by μ_1, μ_2 and μ_3 is given by

$$Z_{\mu_1, \mu_2, \mu_3} = \frac{1}{2\pi i} \oint \frac{\langle \mu_1 | \otimes \langle \mu_2 | \otimes \langle \mu_3 | V \rangle}{\Theta}.$$

5.4. Fermionic gluing principle of the topological vertex - general cases.

We start by assuming the framed ADKMV conjecture is true, and let $V^{(a,b,c)}$ ($a, b, c \in \mathbb{Z}$) be the vectors defined as in the first paragraph of the above subsection.

Let X be a toric Calabi-Yau threefold whose toric diagram is Γ_X . Assume Γ_X has g loops and h vertices and n noncompact edges L_1, \dots, L_n . Let \tilde{Z} be the generating function of the Gromov-Witten invariants of X with framings a_1, \dots, a_n on L_1, \dots, L_n respectively. Denote by Z_{closed} the generating function of the closed Gromov-Witten invariants of X . Then $Z_X = \tilde{Z}/Z_{\text{closed}}$ is the generating function of the open Gromov-Witten invariants of X . As formulated in §5.1, Z_X is a vector in $\Lambda^{\otimes n}$ with parameters, where Λ is the space of symmetric functions. By the boson-fermion correspondence (multi-component version of (9)), Z_X corresponds to a vector, say V_X , in $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_n^{(0)}$.

There are two possible cases that are similar to those described in §5.1:

Case 1. There exists a toric Calabi-Yau threefold X' with toric diagram $\Gamma_{X'}$ such that Γ_X can be constructed by gluing $\Gamma_{X'}$ and the trivalent vertex along a noncompact edge. Then $\Gamma_{X'}$ has g loops and $h - 1$ vertices. Let L_1, \dots, L_{n-2}, L_a be the noncompact edges of $\Gamma_{X'}$, and let $Z_{X'}$ be the generating function of the open Gromov-Witten invariants of X' , with framing a_1, \dots, a_{n-2} on L_1, \dots, L_{n-2} respectively and with canonical framing on L_a . Assume that there is a Bogoliubov transform $V_{X'} \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_{n-2} \otimes \mathcal{F}_a$ whose projection $V_{X',0}$ on $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_{n-2}^{(0)} \otimes \mathcal{F}_a^{(0)}$ is a Laurent polynomial of g formal variables $\Theta_1, \dots, \Theta_g$. Assume the constant term (with respect to $\Theta_1, \dots, \Theta_g$) of $V_{X',0}$ gives the image of $Z_{X'}$ under the boson-fermion correspondence. Provided the framed ADKMV conjecture, we can define the normalized gluing

$$V' = G^f(V_{X'}, V^{(0, a_{n-1}, a_n)})$$

of $V_{X'}$ and $V^{(0, a_{n-1}, a_n)}$ with the special gluing vector $|P_{ab}^f\rangle$ (see §4.2), where f is an integer, and we label the edges of the trivalent vertex by L_b, L_{n-1}, L_n . By Theorem 4.2, V' is a Bogoliubov transform in $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$. By Lemma 4.3 and the gluing procedure as shown in (25), if Γ_X can be constructed by gluing $\Gamma_{X'}$ and the trivalent vertex along the noncompact edges L_a and L_b , then, for a suitable choice of f , the constant term (with respect to $\Theta_1, \dots, \Theta_g$) of the projection of V' on $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_n^{(0)}$ coincides with V_X .

Case 2. There exists a toric Calabi-Yau threefold X'' whose toric diagram $\Gamma_{X''}$ has $n+2$ noncompact edges, say $L_1, \dots, L_n, L_a, L_b$, such that Γ_X can be constructed from $\Gamma_{X''}$ by gluing L_a and L_b . Then $\Gamma_{X''}$ has $g - 1$ loops and h vertices. Let $Z_{X''}$ be the generating function of the open Gromov-Witten invariants of X'' , with

framings a_1, \dots, a_n on L_1, \dots, L_n respectively and with canonical framing on L_a and L_b . Assume that there is a Bogoliubov transform $V_{X''} \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n \otimes \mathcal{F}_a \otimes \mathcal{F}_b$ whose projection $V_{X'',0}$ to $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_n^{(0)} \otimes \mathcal{F}_a^{(0)} \otimes \mathcal{F}_b^{(0)}$ is a Laurent polynomial of $g-1$ formal variables $\Theta_1, \dots, \Theta_{g-1}$. Assume the constant term (with respect to $\Theta_1, \dots, \Theta_{g-1}$) of $V_{X'',0}$ gives the image of $Z_{X''}$ under the boson-fermion correspondence. Provided the framed ADKMV conjecture, we can define the normalized self-gluing

$$V'' = G^f(V_{X''})$$

of $V_{X''}$ with the special gluing vector $\langle P_{ab}^f |$, (see §4.2), where f is an integer. By Theorem 4.2, V'' is a Bogoliubov transform in $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$. Note that V'' contains a new formal variable, say Θ_g , that still appears in the projection V''_0 of V'' on $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_n^{(0)}$. By Lemma 4.3 and the gluing procedure as shown in (26), there exists a suitable integer f such that the constant term (with respect to $\Theta_1, \dots, \Theta_g$) of the projection of V'' to $\mathcal{F}_1^{(0)} \otimes \dots \otimes \mathcal{F}_n^{(0)}$ gives V_X .

By the gluing procedure of the topological vertex as shown in §5.1, the above discussion implies:

Theorem 5.2. *Let X be a toric Calabi-Yau threefold with n outer branes. Assume the generating function of the open Gromov-Witten invariants of X is*

$$Z(\mathbf{x}^1, \dots, \mathbf{x}^n) = \sum_{\mu^1, \dots, \mu^n} Z_{\mu^1 \dots \mu^n} s_{\mu^1}(\mathbf{x}^1) \dots s_{\mu^n}(\mathbf{x}^n).$$

Provided the framed ADKMV conjecture, we have:

1). *if the toric diagram of X has no loops, then there is a Bogoliubov transform $V \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$ such that*

$$Z_{\mu^1 \dots \mu^n} = \langle \mu^1 | \otimes \dots \otimes \langle \mu^n | V \rangle$$

for all partitions μ^1, \dots, μ^n ; and

2). *if the toric diagram of X has g loops, then there is a Bogoliubov transform $V \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$, which is a Laurent series of g formal parameters $\Theta_1, \dots, \Theta_g$, such that*

$$Z_{\mu^1 \dots \mu^n} = \frac{1}{(2\pi i)^g} \oint \frac{\langle \mu^1 | \otimes \dots \otimes \langle \mu^n | V \rangle}{\Theta_1 \dots \Theta_g} d\Theta_1 \dots d\Theta_g$$

for all partitions μ^1, \dots, μ^n .

We refer to Theorem 5.2 as the fermionic gluing principle of the topological vertex.

Remark 5.1. Let $V \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$ be given by 2) in Theorem 5.2. Let

$$Z(\Theta_1, \dots, \Theta_g) = \sum_{\mu^1, \dots, \mu^n} \langle \mu^1 | \otimes \dots \otimes \langle \mu^n | V \rangle s_{\mu^1}(\mathbf{x}^1) \dots s_{\mu^n}(\mathbf{x}^n),$$

which is a Laurent series of formal parameters $\Theta_1, \dots, \Theta_g$ with coefficients in $\Lambda^{\otimes n}$. We can view $Z(\Theta_1, \dots, \Theta_g)$ as an extended topological string partition function of X . Expanding $Z(\Theta_1, \dots, \Theta_g)$ as

$$Z(\Theta_1, \dots, \Theta_g) = \sum_{N_1, \dots, N_g \in \mathbb{Z}} Z_{N_1, \dots, N_g} \Theta_1^{N_1} \dots \Theta_g^{N_g},$$

we have seen that the constant term $Z_{0, \dots, 0}$ gives the generating function of the open Gromov-Witten invariants of X , the toric Calabi-Yau space considered in 2)

in Theorem 5.2. It is natural to consider the geometric meaning of other terms. It was suggested in [2] and [1] from the B-theory viewpoint that Z_{N_1, \dots, N_g} give the partition functions of the Kodaira-Spencer fields on the mirror curve of X with certain monodromy datum represented by (N_1, \dots, N_g) . Surprisingly, it was also argued in the same papers that the additional sectors Z_{N_1, \dots, N_g} contain no more information than $Z_{0, \dots, 0}$. Moreover, a formula that explicitly expresses Z_{N_1, \dots, N_g} in term of $Z_{0, \dots, 0}$ was also proposed, which seems to be another interesting conjecture that asks for further study.

6. PROOF OF THE SELF-GLUING RULE FOR BOGOLIUBOV TRANSFORMS

The aim of this section is to prove Theorem 3.1. To make the key idea clear and concrete, we first consider a simple case of this theorem with full details in §6.1, and then sketch the proof of general cases in §6.2.

6.1. Simple case. Let

$$(29) \quad V_1 = \exp\left(\sum_{i,j=1,2} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) |0_{12}\rangle$$

be a two-component Bogoliubov transform in $\mathcal{F}_1 \otimes \mathcal{F}_2$ and

$$(30) \quad V_2 = \exp\left(\sum_{m,n \geq 0} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle$$

be an one-component Bogoliubov transform in \mathcal{F}_2 , where A_{mn}^{ij} and A_{mn} are arbitrary coefficients maybe with parameters.

Define

$$(31) \quad \tilde{V}_2 = \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle,$$

where Q is a formal variable and \tilde{V}_2 is viewed as a formal power series of Q with coefficients in \mathcal{F}_2 .

The following inner product

$$(32) \quad \begin{aligned} V_0 &= \left(\exp\left(\sum_{m,n \geq 0} A_{mn}^{22} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle, \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle \right) \\ &= \langle 0_2 | \exp\left(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{-\mathbf{n}}^2 \psi_{\mathbf{m}}^{2*}\right) \exp\left(\sum_{m,n \geq 0} A_{mn}^{22} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}\right) |0_2\rangle \end{aligned}$$

is well defined as a formal power series of the formal variable Q .

Define $\tilde{V} = (V_1, \tilde{V}_2)$, where the inner product is taken on the \mathcal{F}_2 component. Then \tilde{V} is a formal power series of Q with coefficients in \mathcal{F}_1 .

Theorem 6.1. *Let \tilde{V} and V_0 be as above, then the formal power series $V = \tilde{V}/V_0$ of Q with coefficients in \mathcal{F}_1 is a Bogoliubov transform of the fermionic vacuum $|0_1\rangle \in \mathcal{F}_1$, i.e., for $m, n \geq 0$, there exist formal power series R_{mn} of Q , such that*

$$(33) \quad V = \exp\left(\sum_{m,n \geq 0} R_{mn} \psi_{\mathbf{m}}^1 \psi_{-\mathbf{n}}^{1*}\right) |0_1\rangle.$$

From the proof of this theorem, we will see why V_0 appear naturally as a factor of \tilde{V} . The following lemma, which is well-known in Lie theory, will be used in the proof.

Lemma 6.2. (See e.g [6]) Let A and B be two linear operators on a vector space H . Assume both e^A and e^B make sense. If the commutator $[A, B] = AB - BA$ commutes with both A and B . Then

$$(34) \quad e^A e^B = e^{[A, B]} e^B e^A.$$

Proof. (Proof of Theorem 6.1) Recall that

$$\tilde{V} = (\exp(\sum_{i,j=1,2} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}) |0_{12}\rangle, \exp(\sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}) |0_2\rangle).$$

To make the notations simpler, let

$$(35) \quad \mathcal{A}^{ij} = \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*},$$

$$(36) \quad \mathcal{A} = \sum_{m,n \geq 0} Q^{m+n+1} A_{mn} \psi_{\mathbf{m}}^2 \psi_{-\mathbf{n}}^{2*}.$$

Then one can rewrite \tilde{V} as follows:

$$(37) \quad \tilde{V} = \exp \mathcal{A}^{11} \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{21}) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) |0_2\rangle |0_1\rangle.$$

As usual, our strategy here is to move the annihilators to the right using the anti-commutation relations (5). By (5) and (11), one has

$$(38) \quad [\mathcal{A}^*, \mathcal{A}^{21}] = \mathcal{B}^{21} = \sum_{m,n \geq 0} B_{mn}^{21} \psi_{-\mathbf{m}}^2 \psi_{-\mathbf{n}}^{1*},$$

where

$$(39) \quad B_{mn}^{21} = \sum_{r \geq 0} Q^{r+m+1} A_{rm} A_{rn}^{21}$$

are formal power series of Q which are divisible by Q . Note that the right-hand side commutes with both \mathcal{A}^* and \mathcal{A}^{21} , hence by Lemma 6.2 we get

$$\exp(\mathcal{A}^*) \exp(\mathcal{A}^{21}) = \exp(\mathcal{A}^{21}) \exp(\mathcal{A}^*) \exp(\mathcal{B}^{21}).$$

By the same method, one can show that

$$(40) \quad \exp(\mathcal{B}^{21}) \exp(\mathcal{A}^{12}) = \exp(\mathcal{A}^{2112,1}) \exp(\mathcal{A}^{12}) \exp(\mathcal{B}^{21}),$$

where

$$(41) \quad \mathcal{A}^{2112,1} = [\mathcal{B}^{21}, \mathcal{A}^{12}] = - \sum_{m,n \geq 0} \sum_{r \geq 0} A_{mr}^{12} B_{rn}^{21} \psi_m^1 \psi_{-n}^{1*},$$

and

$$(42) \quad \exp(\mathcal{B}^{21}) \exp(\mathcal{A}^{22}) = \exp(\mathcal{A}^{21,1}) \exp(\mathcal{A}^{22}) \exp(\mathcal{B}^{21}),$$

where

$$(43) \quad \mathcal{A}^{21,1} = [\mathcal{B}^{21}, \mathcal{A}^{22}] = - \sum_{m,n \geq 0} \sum_{r \geq 0} A_{mr}^{22} B_{rn}^{21} \psi_m^2 \psi_{-n}^{1*}.$$

Now we have

$$\tilde{V} = \exp(\mathcal{A}^{11} + \mathcal{A}^{2112,1}) \langle 0_2 | \exp(\mathcal{A}^{21}) \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{B}^{21}) |0_2\rangle |0_1\rangle.$$

Note $\mathcal{B}^{21} |0_2\rangle = 0$ and $\langle 0_2 | \mathcal{A}^{21} = 0$, we have

$$(44) \quad \tilde{V} = \exp(\mathcal{A}^{11} + \mathcal{A}^{2112,1}) \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) |0_2\rangle |0_1\rangle.$$

Similarly, one can show that

$$(45) \quad \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) = \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^*) \exp(\mathcal{B}^{12}),$$

where

$$\mathcal{B}^{12} = [\mathcal{A}^*, \mathcal{A}^{12}] = \sum_{m,n \geq 0} B_{mn}^{12} \psi_{\mathbf{m}}^1 \psi_{\mathbf{n}}^{2*}$$

which commutes with both \mathcal{A}^* and \mathcal{A}^{12} , where

$$(46) \quad B_{mn}^{12} = - \sum_{r \geq 0} A_{mr}^{12} Q^{n+r+1} A_{nr}$$

are formal power series of Q which are divisible by Q .

$$(47) \quad \exp(\mathcal{B}^{12}) \exp(\mathcal{A}^{22}) = \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{B}^{12}),$$

where

$$(48) \quad \mathcal{A}^{12,1} = [\mathcal{B}^{12}, \mathcal{A}^{22}] = \sum_{m,n \geq 0} \sum_{r \geq 0} B_{mr}^{12} A_{rn}^{22} \psi_{\mathbf{m}}^1 \psi_{-\mathbf{n}}^{2*},$$

and

$$(49) \quad \exp(\mathcal{B}^{12}) \exp(\mathcal{A}^{21,1}) = \exp(\mathcal{A}^{1221,1}) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{B}^{12}),$$

where

$$(50) \quad \mathcal{A}^{1221,1} = [\mathcal{B}^{12}, \mathcal{A}^{21,1}] = \sum_{m,n \geq 0} \left(\sum_{r \geq 0} B_{mr}^{12} A_{rn}^{21,1} \right) \psi_{\mathbf{m}}^1 \psi_{-\mathbf{n}}^{1*}$$

commutes with both \mathcal{B}^{12} and $\mathcal{A}^{21,1}$. Because $\langle 0_2 | \mathcal{A}^{12} = 0$ and $\mathcal{B}^{12} | 0_2 \rangle = 0$,

$$\begin{aligned} & \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle \\ &= \langle 0_2 | \exp(\mathcal{A}^{12}) \exp(\mathcal{A}^*) \exp(\mathcal{B}^{12}) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle \\ &= \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{A}^{1221,1}) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{B}^{12}) | 0_2 \rangle \\ &= \exp(\mathcal{A}^{1221,1}) \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{22}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{A}^{21,1}) | 0_2 \rangle. \end{aligned}$$

Because the operators \mathcal{A}^{22} , $\mathcal{A}^{12,1}$ and $\mathcal{A}^{21,1}$ commute with each other, we now have

$$\tilde{V} = \exp(\mathcal{A}^{11} + \mathcal{A}^{2112,1} + \mathcal{A}^{1221,1}) \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{21,1}) \exp(\mathcal{A}^{12,1}) \exp(\mathcal{A}^{22}) | 0_2 \rangle | 0_1 \rangle.$$

Recall $\mathcal{A}^{2112,1}$, $\mathcal{A}^{12,1}$, $\mathcal{A}^{21,1}$ are divisible by Q , and $\mathcal{A}^{1221,1}$ is divisible by Q^2 . By repeating the above procedure N -times one gets:

$$\begin{aligned} \tilde{V} &= \exp(\mathcal{A}^{11} + \sum_{j=1}^N (\mathcal{A}^{2112,j} + \mathcal{A}^{1221,j})) \\ & \quad \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{21,N}) \exp(\mathcal{A}^{12,N}) \exp(\mathcal{A}^{22}) | 0_2 \rangle | 0_1 \rangle, \end{aligned}$$

where $\mathcal{A}^{2112,j}$, $\mathcal{A}^{12,j}$, $\mathcal{A}^{21,j}$ and $\mathcal{A}^{1221,j}$ is divisible by Q^j . Therefore, by taking $N \rightarrow \infty$,

$$(51) \quad \tilde{V} = \langle 0_2 | \exp(\mathcal{A}^*) \exp(\mathcal{A}^{22}) | 0_2 \rangle \cdot \exp(\mathcal{A}^{11} + \sum_{j=1}^{\infty} (\mathcal{A}^{2112,j} + \mathcal{A}^{1221,j})) | 0_1 \rangle.$$

This completes the proof of Theorem 4.1. \square

6.2. Sketch of the proof of Theorem 3.1. The proof of Theorem 3.1 is essentially same as that of Theorem 6.1, but with much more complicated calculations. So we just sketch it here.

Proof. (Sketch of the Proof of theorem 3.1) Recall that the gluing vector $P_{ab}^{\mathbf{E}}$ is defined in (13). We write

$$V = \exp \left(\sum_{i,j \in \{a,b,1,2,\dots,M\}} \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*} \right) |0\rangle.$$

The self-gluing $\tilde{G}^{\mathbf{E}}(V)$ is given by (14). For $r \in \mathbb{Z} + 1/2$, the adjoint operator of ψ_r^i is ψ_r^{i*} . Let

$$\begin{aligned} \mathcal{A}_{ij} &= \sum_{m,n \geq 0} A_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}, \quad i, j \in \{1, \dots, n, a, b\}, \\ \mathcal{B}_{ij} &= \sum_{m,n \geq 0} B_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}, \quad i, j = a, b, \end{aligned}$$

where $B_{mn}^{ij} = Q^{m+n+1} \Theta^{\delta_{ij}^{ab}} E_{mn}^{ij}$ for $i, j = a, b$. Then

$$\begin{aligned} \tilde{G}^{\mathbf{E}}(V) &= \exp \left(\sum_{i,j=1}^M \mathcal{A}_{ij} \right) \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \\ &\quad \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \\ &\quad \prod_{i=1}^M \exp(\mathcal{A}_{ai}) \exp(\mathcal{A}_{bi}) \prod_{i=1}^M \exp(\mathcal{A}_{ia}) \exp(\mathcal{A}_{ib}) |0\rangle |0'\rangle, \end{aligned} \quad (52)$$

where $|0\rangle$ is the vacuum in $\mathcal{F}_a \otimes \mathcal{F}_b$ and $|0'\rangle$ is the vacuum in $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_M$.

Note that

$$\langle 0 | \exp(\mathcal{A}_{ip}) = \langle 0 | \exp(\mathcal{A}_{pi}) = \langle 0 |$$

for $i = 1, \dots, M$, $p = a, b$, so we try to move these operators to the left. We first consider $\exp(\mathcal{A}_{ia})$. By (5) and (11), for a fixed i , $1 \leq i \leq M$, we have

$$[\mathcal{B}_{aa}^*, \mathcal{A}_{ia}] = \mathcal{S}_{ia} = \sum_{m,n \geq 0} S_{mn}^{ia} \psi_{\mathbf{m}}^i \psi_{\mathbf{n}}^{a*}, \quad (53)$$

where

$$S_{mn}^{ia} = \sum_{r \geq 0} A_{mr}^{ia} B_{nr}^{aa}$$

are formal power series of Q , Θ and Θ^{-1} . It is important for our argument that the coefficients S_{mn} are divisible by Q . It is clear that \mathcal{S}_{ia} commutes with both B_{aa}^* and \mathcal{A}_{ia} . By Lemma 6.2, we have

$$\exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{ia}) = \exp(\mathcal{A}_{ia}) \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{S}_{ia}). \quad (54)$$

Similarly, one can prove that

$$\exp(\mathcal{B}_{ba}^*) \exp(\mathcal{A}_{ia}) = \exp(\mathcal{A}_{ia}) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{S}_{ib}), \quad (55)$$

where

$$\mathcal{S}_{ib} = \sum_{m,n \geq 0} S_{mn}^{ib} \psi_{\mathbf{m}}^i \psi_{\mathbf{n}}^{b*}$$

is an operator divisible by Q .

Note that $\langle 0 | \psi_{-\mathbf{n}}^{a*} = 0$ for all $n \geq 0$, we have

$$(56) \quad \begin{aligned} & \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{ia}) \\ &= \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{S}_{ib}) \exp(\mathcal{S}_{ia}). \end{aligned}$$

Now we get two new operators $\exp \mathcal{S}_{ia}$ and $\exp \mathcal{S}_{ib}$ that seems unrelated to the original representation of $\tilde{G}^E(V)$ in (52). Note that $\exp \mathcal{S}_{ia} |0\rangle = \exp \mathcal{S}_{ib} |0\rangle = |0\rangle$, we can get ride of these operators by moving them to the right. By the same argument, one can show that

$$(57) \quad \begin{aligned} & \exp(\mathcal{S}_{ib}) \exp(\mathcal{S}_{ia}) \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \\ &= \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \exp(\mathcal{A}'_{ia,1}) \exp(\mathcal{A}'_{ib,1}) \exp(\mathcal{S}_{ib}) \exp(\mathcal{S}_{ia}), \end{aligned}$$

where

$$\mathcal{A}'_{ia,1} = \sum_{mn \geq 0} A_{mn}^{ia,1} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{a*}, \quad \mathcal{A}'_{ib,1} = \sum_{m,n \geq 0} A_{mn}^{ib,1} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{b*}$$

are certain operators divisible by Q . Finally we have the commutation relation

$$(58) \quad \begin{aligned} & \exp(\mathcal{S}_{ib}) \exp(\mathcal{S}_{ia}) \prod_{j=1}^M \exp(\mathcal{A}_{aj}) \exp(\mathcal{A}_{bj}) \\ &= \exp\left(\sum_{j=1}^M \mathcal{A}'_{ij,1}\right) \prod_{j=1}^M \exp(\mathcal{A}_{aj}) \exp(\mathcal{A}_{bj}) \exp(\mathcal{S}_{ib}) \exp(\mathcal{S}_{ia}), \end{aligned}$$

where, for $j = 1, \dots, M$,

$$\mathcal{A}'_{ij,1} = \sum_{m,n \geq 0} A_{mn}^{ij,1} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}$$

are certain operators divisible by Q . By (56), (57), (58) and (52), we get

$$(59) \quad \begin{aligned} \tilde{G}^E(V) &= \exp\left(\sum_{i,j=1}^M (\mathcal{A}_{ij} + \mathcal{A}'_{ij,1})\right) \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \exp(\mathcal{B}_{aa}^*) \\ & \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \prod_{i=1}^M \exp(\mathcal{A}_{ai}) \exp(\mathcal{A}_{bi}) \\ & \prod_{i=1}^M \exp(\mathcal{A}_{ib}) \exp\left(\sum_{i=1}^M \mathcal{A}'_{ia,1}\right) \exp\left(\sum_{i=1}^M \mathcal{A}'_{ib,1}\right) |0\rangle |0'\rangle. \end{aligned}$$

Repeating the above process for the operator $\prod_{i=1}^M \exp(\mathcal{A}_{ib})$, we see that, for $i, j = 1, \dots, M$, there are operators

$$\begin{aligned} \mathcal{A}'_{ij,2} &= \sum_{m,n \geq 0} A_{mn}^{ij,2} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}, \\ \mathcal{A}'_{ia,2} &= \sum_{m,n \geq 0} A_{mn}^{ia,2} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{a*}, \\ \mathcal{A}'_{ib,2} &= \sum_{m,n \geq 0} A_{mn}^{ib,2} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{b*} \end{aligned}$$

divisible by Q such that

$$\begin{aligned}
& \tilde{G}^{\mathbf{E}}(V) \\
&= \exp\left(\sum_{i,j=1}^M (\mathcal{A}_{ij} + \mathcal{A}'_{ij})\right) \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \\
(60) \quad & \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \prod_{i=1}^M \exp(\mathcal{A}_{ai}) \exp(\mathcal{A}_{bi}) \\
& \exp\left(\sum_{i=1}^M \mathcal{A}_{ia}(1)\right) \exp\left(\sum_{i=1}^M \mathcal{A}_{ib}(1)\right) |0\rangle |0'\rangle,
\end{aligned}$$

where $\mathcal{A}'_{ij} = \mathcal{A}'_{ij,1} + \mathcal{A}'_{ij,2}$, $\mathcal{A}_{ia}(1) = \mathcal{A}'_{ia,1} + \mathcal{A}'_{ia,2}$, $\mathcal{A}_{ib}(1) = \mathcal{A}'_{ib,1} + \mathcal{A}'_{ib,2}$.

We can carry out similar process for the operators $\prod_{i=1}^M \exp(\mathcal{A}_{ai})$ and $\prod_{i=1}^M \exp(\mathcal{A}_{bi})$ to show that, for $i = 1, \dots, M$, there are operators

$$\begin{aligned}
\mathcal{A}''_{ij} &= \sum_{m,n \geq 0} A''_{mn}{}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}, \\
\mathcal{A}_{ai}(1) &= \sum_{m,n \geq 0} A_{mn}^{ai}(1) \psi_{\mathbf{m}}^a \psi_{-\mathbf{n}}^{i*}, \\
\mathcal{A}_{bi}(1) &= \sum_{m,n \geq 0} A_{mn}^{bi}(1) \psi_{\mathbf{m}}^a \psi_{-\mathbf{n}}^{i*}
\end{aligned}$$

divisible by Q such that

$$\begin{aligned}
\tilde{G}^{\mathbf{E}}(V) &= \exp\left(\sum_{i,j=1}^M (\mathcal{A}_{ij} + \mathcal{A}_{ij}(1))\right) \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \\
(61) \quad & \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \\
& \prod_{i=1}^M \exp(\mathcal{A}_{ai}(1)) \exp(\mathcal{A}_{bi}(1)) \prod_{i=1}^M \exp(\mathcal{A}_{ia}(1)) \exp(\mathcal{A}_{ib}(1)) |0\rangle |0'\rangle,
\end{aligned}$$

where $\mathcal{A}_{ij}(1) = \mathcal{A}'_{ij} + \mathcal{A}''_{ij}$.

At the first glance, it seems that we go back to the starting point (52) and get nothing helpful. But it is not the case. The key point here is that all the operators $\mathcal{A}_{ij}(1)$, $\mathcal{A}_{ia}(1)$, $\mathcal{A}_{ib}(1)$, $\mathcal{A}_{ai}(1)$ and $\mathcal{A}_{bi}(1)$, when viewed as formal power series of Q , are divisible by Q . Repeat the above process inductively, for $N = 1, 2, \dots$, we get a series of operators $\mathcal{A}_{ij}(N)$, $\mathcal{A}_{ia}(N)$, $\mathcal{A}_{ib}(N)$, $\mathcal{A}_{ai}(N)$ and $\mathcal{A}_{bi}(N)$ which are formal power series of Q , Θ and Θ^{-1} and divisible by Q^N , such that

$$\begin{aligned}
\tilde{G}^{\mathbf{E}}(V) &= \exp\left(\sum_{i,j=1}^M \left(\sum_{k=0}^{N+1} \mathcal{A}_{ij}(k)\right)\right) \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \\
(62) \quad & \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) \\
& \prod_{i=1}^M \exp(\mathcal{A}_{ai}(N+1)) \exp(\mathcal{A}_{bi}(N+1)) \\
& \prod_{i=1}^M \exp(\mathcal{A}_{ia}(N+1)) \exp(\mathcal{A}_{ib}(N+1)) |0\rangle |0'\rangle,
\end{aligned}$$

for each integer $N > 0$, where we set $\mathcal{A}_{ij}(0) = \mathcal{A}_{ij}$.

For a formal power series $f = \sum_{n \geq 0} f_n Q^n$ of Q with coefficients in $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$ and a positive integer N , we denote by $[f]_N$ the sum $\sum_{n=0}^N f_n Q^n$. Note that by definition

$$(63) \quad \tilde{G}^{\mathbf{E}}(V)_{closed} = \langle 0 | \exp(\mathcal{B}_{bb}^*) \exp(\mathcal{B}_{ba}^*) \exp(\mathcal{B}_{ab}^*) \exp(\mathcal{B}_{aa}^*) \exp(\mathcal{A}_{aa}) \exp(\mathcal{A}_{ab}) \exp(\mathcal{A}_{ba}) \exp(\mathcal{A}_{bb}) | 0 \rangle.$$

By (62), for each positive integer N , we have

$$(64) \quad [\tilde{G}^{\mathbf{E}}(V)]_N = \left[\tilde{G}^{\mathbf{E}}(V)_{closed} \exp\left(\sum_{i,j=1}^M \left(\sum_{k=0}^{N+1} \mathcal{A}_{ij}(k)\right)\right) | 0' \rangle \right]_N.$$

For $i, j = 1, \dots, M$, we define

$$(65) \quad \mathcal{R}_{ij} = \sum_{k \geq 0} \mathcal{A}_{ij}(k).$$

Since $\mathcal{A}_{ij}(k)$ are divisible by Q^k , $k > 0$, \mathcal{R}_{ij} are well defined as formal power series of Q and have the form

$$\mathcal{R}_{ij} = \sum_{m,n \geq 0} R_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}.$$

By (64), we have

$$(66) \quad [\tilde{G}^{\mathbf{E}}(V)]_N = \left[\tilde{G}^{\mathbf{E}}(V)_{closed} \exp\left(\sum_{i,j=1}^M \mathcal{R}_{ij}\right) | 0' \rangle \right]_N$$

for all positive integer N . So we have

$$(67) \quad \tilde{G}^{\mathbf{E}}(V) = \tilde{G}^{\mathbf{E}}(V)_{closed} \exp\left(\sum_{i,j=1}^M \sum_{m,n \geq 0} R_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) | 0' \rangle,$$

and hence

$$(68) \quad G^{\mathbf{E}}(V) = \exp\left(\sum_{i,j=1}^M \sum_{m,n \geq 0} R_{mn}^{ij} \psi_{\mathbf{m}}^i \psi_{-\mathbf{n}}^{j*}\right) | 0' \rangle$$

is a Bogoliubov transform. \square

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